L. 14 and 15

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Decomposition of Riemann

$$\boxed{R^{\mu\nu}{}_{\rho\sigma} = C^{\mu\nu}{}_{\rho\sigma} + \frac{4}{n-2}\,\delta^{[\mu}_{[\rho}\,\check{R}^{\nu]}{}_{\sigma]} + \frac{2}{n(n-1)}\,R\,\delta^{\mu}_{[\rho}\delta^{\nu}_{\sigma]}}$$

$\boxed{n \geq 3}$

this, when solved for $C^{\mu\nu}{}_{\rho\sigma}$ gives a definition of Weyl.

$$\Sigma : \begin{cases} k\, x^{0\,2} + g_{\mu\nu} x^\mu x^\nu = k r^2 \\ K = \frac{1}{k r^2} \end{cases} ; \quad (x^0, x^\mu) \in \mathbb{R} \times \mathbb{R}^n$$

$$\tilde{g} = k\, dx^{0\,2} + g_{\mu\nu} dx^\mu dx^\nu$$

$$y^\mu = \frac{2x^\mu}{1 - \frac{x^0}{r}}$$

Canonical metric on a hyperquadric Σ: $\Rightarrow$

$$\boxed{\tilde{g}|_\Sigma = \frac{g_{\mu\nu}\, dy^\mu dy^\nu}{\left(1 + \frac{K}{4} g_{\mu\nu} y^\mu y^\nu\right)^2}} = g_{\mu\nu}\,\omega^\mu \omega^\nu$$

Its curvature is

$$\boxed{R^{\mu\nu}{}_{\rho\sigma} = 2K\, \delta^{\mu}_{[\rho}\delta^{\nu}_{\sigma]}}$$ in the coframe $\omega^\mu = \dfrac{dy^\mu}{1 + \frac{K}{4} g_{\mu\nu} y^\mu y^\nu}$

$$\Rightarrow \quad C^{\mu\nu}{}_{\rho\sigma} \equiv 0, \quad \check{R}^{\nu}{}_{\sigma} \equiv 0, \quad \frac{R}{n(n-1)} = K \quad \Rightarrow \boxed{R = n(n-1)K}$$

Since $C^{\mu\nu}{}_{\rho\sigma}$, $\check{R}^{\nu}{}_{\sigma}$ vanish and K is const, this spaces have maximal group of symmetries (otherwise the group should preserve $\check{R}^{\nu}{}_{\sigma}$, $C^{\mu\nu}{}_{\rho\sigma}$ and would be reduced.)

(M,g) is Einstein $\equiv$ $\boxed{\begin{array}{c} \check{R}_{\mu\nu} \equiv 0 \\ \Updownarrow \text{Thm} \\ R_{\mu\nu} = \tilde{\Lambda} g_{\mu\nu} \\ \text{and } \tilde{\Lambda} = \text{const} \end{array}}$ or

$$\check{R}_{\mu\nu} = 0 \iff R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu} = 0 \iff$$

$$R_{\mu\nu} = \frac{1}{n} R g_{\mu\nu}$$

note that 2^{nd} Bianchi ietentity gives:

$$\nabla_{[\mu} R^{\alpha}{}_{|\beta|\gamma\delta]} = 0$$

$$\nabla_\mu R^\alpha{}_{\beta\gamma\delta} + \nabla_\delta R^\alpha{}_{\beta\mu\gamma} + \nabla_\gamma R^\alpha{}_{\beta\delta\mu} = 0 \qquad |_{\mu \nearrow \alpha}$$

$$\nabla_\mu R^\mu{}_{\beta\gamma\delta} + \nabla_\delta R_{\beta\gamma} - \nabla_\gamma R_{\beta\delta} = 0 \qquad |_{\beta \nearrow \delta}$$

$$\nabla_\mu R^\mu{}_\gamma + \nabla_\mu R^\mu{}_\gamma - \nabla_\gamma R = 0.$$

$$\boxed{\nabla^\mu \left(R_{\mu\gamma} - \frac{1}{2} g_{\mu\gamma} R \right) = 0}$$

Key identity in the theory of Relativity

$$\boxed{G_{\mu\gamma} = R_{\mu\gamma} - \frac{1}{2} g_{\mu\gamma} R}$$ $\leftarrow$ Einstein tensor.

If (M,g) is Einstein, then

$$\frac{1}{2} \nabla_\nu R = \nabla^\mu R_{\mu\nu} = \frac{1}{n} \nabla^\mu (R g_{\mu\nu}) = \frac{1}{n} \nabla_\nu R \qquad n \neq 2$$

$$\Rightarrow \nabla_\nu R = 0 \Rightarrow R = \text{const}.$$

$$R_{\mu\nu} = \tilde{\Lambda} g_{\mu\nu} \Rightarrow R = n\tilde{\Lambda} \Rightarrow \tilde{\Lambda} = \frac{1}{n} R \Rightarrow \check{R}_{\mu\nu} = 0.$$

$\square$.

Einstein equations

$$\boxed{G_{\mu\nu} = T_{\mu\nu}} \leftarrow \text{energy momentum of matter fields.}$$

Bianchi identity $\nabla^{\mu} G_{\mu\nu} \equiv 0$ gives

$$\nabla^{\mu} T_{\mu\nu} \equiv 0 \quad \text{— conservation of energy.}$$

Einstein theory

(M, g), $\dim M = 4$, $p = 1$, $q = 3$

g satisfies $\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = T_{\mu\nu}}$

Einstein equations with cosmological constant.

cosmological constant. (dark energy)

Example of solutions:

take Σ_1 when $n = 4$, $g_{\mu\nu}$ of sign. $p = 1$, $q = 3$.

$$0 = \check{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R = G_{\mu\nu} + \frac{1}{4} R g_{\mu\nu}$$

$$= G_{\mu\nu} + 3K g_{\mu\nu}$$

$$\Lambda = 3K, \quad T_{\mu\nu} = 0.$$

$K = 0 \Rightarrow$ Minkowski space-time; $K \gtrless 0$ De Sitter spacetime / anti De Sitter spacetime.

Axioms for the Lie derivative

$X \in \mathfrak{X}(M)$

$$\mathcal{L}_X : \mathcal{T}(M) \longrightarrow \mathcal{T}(M)$$

such that

1) $\mathcal{L}_X$ is $\mathbb{R}$-linear
2) $\mathcal{L}_X$ preserves the type of tensor
3) $\mathcal{L}_X(K \otimes L) = \mathcal{L}_X K \otimes L + K \otimes \mathcal{L}_X L$
4) $\mathcal{L}_X$ commutes with contractions
5) $\mathcal{L}_X$ commutes with Alt.
6) on forms: • $\mathcal{L}_X$ is a derivation of degree 0.
   $\mathcal{L}_X(\omega \wedge \tilde{\omega}) = \mathcal{L}_X \omega \wedge \tilde{\omega} + \omega \wedge \mathcal{L}_X \tilde{\omega}$
   • $\mathcal{L}_X \alpha = X \lrcorner d\alpha + d(X \lrcorner \alpha)$
   • $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$

   in particular on functions: $\mathcal{L}_X f = X(f)$.
7) $\mathcal{L}_{X_1} \circ \mathcal{L}_{X_2} - \mathcal{L}_{X_2} \circ \mathcal{L}_{X_1} = \mathcal{L}_{[X_1, X_2]}$.

Isometries ; Killing equation

$\varphi : (M,g) \xrightarrow{\text{diff}} (M,g)$ s.t.

$\varphi^* g = g$ is called an isometry of (M,g).

φ_1, φ_2 two isometries, $\varphi_1 \circ \varphi_2$ is also an isometry, as well as φ_1^{-1}.

$\uparrow$

they form an isometry group G of (M,g).

(local) 1-parameter group of isometries

$\varphi_t : (M,g) \longrightarrow (M,g)$ isometries s.t.

$\varphi_t \circ \varphi_{t'} = \varphi_{t+t'}$

We have

$$\varphi_t^* g = g \Rightarrow \mathcal{L}_X g = 0 \text{ where}$$

X a vector field with flow φ_t.

X_1, X_2 s.t. $\mathcal{L}_{X_1} g = 0 = \mathcal{L}_{X_2} g = 0 \Rightarrow \mathcal{L}_{[X_1,X_2]} g = 0$

since $\mathcal{L}_{[X_1,X_2]} = \mathcal{L}_{X_1} \circ \mathcal{L}_{X_2} - \mathcal{L}_{X_2} \circ \mathcal{L}_{X_1}$.

X s.t. $\mathcal{L}_X g = 0$ is called infinitesimal symmetry for g or Killing field.

$\boxed{\mathcal{L}_X g = 0}$ $\leftarrow$ Killing equation

Example

$$g = g_{\mu\nu}\, dx^\mu dx^\nu \qquad g_{\mu\nu} = \mathrm{diag}(1, -1, -1, \ldots -1)$$

$$X = a^\rho \partial_\rho$$

$$\underset{X}{\mathcal{L}} g = \cancel{X(g_{\mu\nu})}\, dx^\mu dx^\nu + 2 g_{\mu\nu}\, \underset{X}{\mathcal{L}}(dx^\mu)\, dx^\nu =$$

$$= 2 g_{\mu\nu}\, d\, \underset{X}{\mathcal{L}}(x^\mu)\, dx^\nu = 2 g_{\mu\nu}\, d(a^\mu)\, dx^\nu =$$

$$= 2 g_{\mu\nu}\, a^\mu{}_{,\rho}\, dx^\rho dx^\nu =$$

$$= 2\, a_{\nu,\rho}\, dx^\rho dx^\nu = 0$$

$$a_{(\nu,\rho)} = 0 \qquad \text{or}$$

$$\partial_{(\rho} a_{\nu)} = 0$$

$$\Rightarrow \begin{cases} \partial_\sigma \partial_{(\rho} a_{\nu)} = 0 \\ \partial_{[\sigma} \partial_{\rho]} a_\nu = 0 \end{cases}$$

$$\Rightarrow \partial_\sigma \partial_\rho a_\nu = 0 \Rightarrow$$

$$\partial_\rho a_\nu = B_{\rho\nu} = \text{const}$$

$$\partial_{(\rho} a_{\nu)} = 0 \Rightarrow \boxed{B_{\rho\nu} = -B_{\nu\rho}}$$

$$\boxed{a_\nu = B_{\rho\nu} x^\rho + C_\nu}$$

Lorentz transf ↑ ↑ translations

$$G = O(g) \ltimes \mathbb{R}^n$$

$$\dim G = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$$

Maximal symmetry group $\Rightarrow$

$$C^{\mu}{}_{\nu\rho\sigma} \equiv 0, \quad \check{R}^{\mu}{}_{\nu} = 0$$

$$\Rightarrow \quad R^{\mu\nu}{}_{\rho\sigma} = \underbrace{\frac{R}{n(n-1)}}_{K} \left(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}\right)$$

$$\Omega^{\mu\nu} = K\,\theta^{\mu}\wedge\theta^{\nu}$$

Bianchi identity:

$$0 = D\Omega^{\mu\nu} = DK\wedge\theta^{\mu}\wedge\theta^{\nu} = dK\wedge\theta^{\mu}\wedge\theta^{\nu}$$

and if $n \geq 3 \Rightarrow \underline{K = \text{const}}$

Can we find all such (M, g)?

First Cartan structure eqs:

$$\begin{cases} d\theta^{\mu} + \Gamma^{\mu}{}_{\nu}\wedge\theta^{\nu} = 0 & \text{(A)} \\ d\Gamma^{\mu\nu} + \Gamma^{\mu}{}_{\rho}\wedge\Gamma^{\rho\nu} = K\,\theta^{\mu}\wedge\theta^{\nu} & \text{(B)} \end{cases} \qquad \underline{\underline{g_{\mu\nu}}}$$

these are satisfied on $\underline{M}$ so $\Gamma^{\mu}{}_{\rho}$ are linearly dependent on (θ^{μ}).

Trick: consider (A) (B) on an abstract $n + \frac{n(n-1)}{2}$ dimensional mfd P where θ^{μ} and $\Gamma^{\mu}{}_{\nu}$ are linearly independent.

assume that we have P

Set. $\dim P = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$

and $\sigma^A = (\underbrace{\theta^\mu}_{n}, \underbrace{\Gamma^\mu{}_\nu}_{\frac{n(n-1)}{2}})$ is a coframe on P

satisfying (A) and (B)

note that if σ^A satisfies (A) and (B) then

$$d\sigma^A = -\frac{1}{2} C^A{}_{BC}\, \sigma^B \wedge \sigma^C$$

where $C^A{}_{BC}$ are <u>all</u> constants.

$\Rightarrow$ $\boxed{C^A{}_{BC} = -C^A{}_{CB}}$ and

$d^2\sigma^A \equiv 0$ is equivalent to

$$\frac{1}{2}\left(C^A{}_{BC}\, C^B{}_{DE}\, \sigma^D \wedge \sigma^E \wedge \sigma^C - C^A{}_{BC}\, C^C{}_{DE}\, \sigma^B \wedge \sigma^D \wedge \sigma^E\right) = 0$$

$$\boxed{C^A{}_{B[C}\, C^B{}_{DE]} = 0}$$

$$[X_A, X_B] = C^C{}_{AB} X_C$$

$$[[X_A, X_B], X_C] + [[X_C, X_A], X_B] + [[X_B, X_C], X_A] =$$

$$C^E{}_{AB}\, C^D{}_{EC} + C^E{}_{CA}\, C^D{}_{EB} + C^E{}_{BC}\, C^D{}_{EA} = C^D{}_{EC}\, C^E{}_{AB} + C^D{}_{EB}\, C^E{}_{CA} +$$

$$+ C^D{}_{EA}\, C^E{}_{BC} = 2\, C^D{}_{E[C}\, C^E{}_{AB]}$$

$$= 0$$

Thus $c^A{}_{BC}$ are structure constants of a certain Lie algebra $\mathfrak{g}$ of dimension $\frac{n(n+1)}{2}$.

Lie algebra is totally determined by $c^A{}_{BC}$. Hence by K.

$$\text{If} \quad K \begin{cases} >0 & \text{all such } \mathfrak{g} \text{ are isomorphic to } \underline{o}(p+1,q) \\ =0 & \text{all such } \mathfrak{g} \text{ are isomorphic to } \underline{o}(p,q)\oplus \mathbb{R}^n \\ <0 & \text{all such } \mathfrak{g} \text{ are isomorphic to } \underline{o}(p,q+1) \end{cases}$$

where $p+q=n$, and g has sign. $(\underset{\text{plus}}{p}, \underset{\text{minus.}}{q})$

P is locally a Lie group $G = \begin{cases} SO(p+1,q) & K>0 \\ SO(p,q) \rtimes \mathbb{R}^n & K=0 \\ SO(p,q+1) & K<0 \end{cases}$

and σ^A are left invariant forms on G.

Left invariant forms on Lie groups are easy to find so we have <u>all</u> solutions to (A), (B) on $P=G$.

How to reconstruct (M, g) having $P = G$?

Maurer-Cartan form

$$\theta_{MC} = g^{-1} dg \qquad g \in G$$

is left invariant, and decomposing it onto the basis of Lie algebra $\mathfrak{g}$ of G $(e_\mu, e_{\alpha\beta})$ we have

$$\theta_{MC} = g^{-1} dg = \theta^\mu e_\mu + \Gamma^{\mu\nu} e_{\mu\nu}$$

which gives us

$$\sigma^A = (\theta^\mu, \Gamma^{\mu\nu}) \quad \text{which solve (A), (B) on } P = G$$

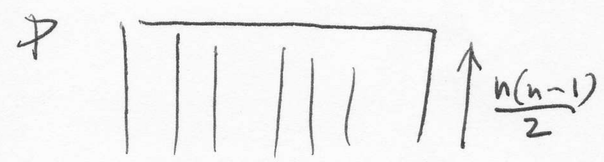
<u>Now</u>: Let $(X_\mu, Y_{\mu\nu})$ be a basis of vector fields on P dual to $(\theta^\mu, \Gamma^{\mu\nu})$.

Observe that

$$d\theta^\mu \wedge \theta^1 \wedge \dots \wedge \theta^n = 0 \qquad \forall \mu = 1, \dots n$$

$\Rightarrow$ Frobenius theorem says that

$G = P$ is foliated by the leaves of INTEGRBLE distribution spanned by $Y_{\mu\nu}$



Consider

$$\hat{g} = g_{\mu\nu} \theta^\mu \theta^\nu \quad \text{where}$$

$$g_{\mu\nu} = \mathrm{diag}(\underbrace{1, \dots 1}_{p}, \underbrace{-1, \dots -1}_{q}).$$

This is a <u>degenerate</u> symmetric bilinear form on P and degeneracy occurs precisely in $\frac{n(n-1)}{2}$ directions spanned by $Y_{\mu\nu}$.

$$\mathcal{L}_{Y_{\mu\nu}} \tilde{g} = ?$$

$$\mathcal{L}_{Y_{\mu\nu}} \left(g_{\alpha\beta}\theta^\alpha\theta^\beta\right) = 2 g_{\alpha\beta}\theta^\alpha \mathcal{L}_{Y_{\mu\nu}}\theta^\beta$$

$$\mathcal{L}_{Y_{\mu\nu}}\theta^\beta = Y_{\mu\nu} \lrcorner d\theta^\beta + d(\cancel{Y_{\mu\nu}\lrcorner\theta^\beta})$$

$$\overset{(A)}{=} - Y_{\mu\nu}\lrcorner\left(\Gamma^{\beta\rho}\wedge\theta_\rho\right) = \frac{1}{2} Y_{\mu\nu}\lrcorner\left[\left(\Gamma^{\rho\beta}-\Gamma^{\beta\rho}\right)\wedge\theta_\rho\right] =$$

$$= \frac{1}{2}\left(\delta^\rho_\mu\delta^\beta_\nu - \delta^\beta_\mu\delta^\rho_\nu\right)\theta_\rho =$$

$$= \delta^{[\rho}_\mu\delta^{\beta]}_\nu\theta_\rho$$

$$\mathcal{L}_{Y_{\mu\nu}}\left(g_{\alpha\beta}\theta^\alpha\theta^\beta\right) = 2\,\theta_\beta\theta_\rho\;\delta^{[\rho}_\mu\delta^{\beta]}_\nu = 0.$$

symmetric.

Thus $\tilde{g}$ descends to the leaf space of the foliation $M = P/\sim$. and g is nondegenerate there.

$$M \overset{\iota}{\hookrightarrow} P, \qquad g = \iota^*\tilde{g}$$

$\Rightarrow$ g satisfies (A) (B) on M. !